Sub-Computabilities

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1. motivation

Absoluteness arguments for Turing degrees

(INCOMP):

“$\exists x, y$ s.t. $x \nleq_T y \;\&\; x \ngeq_T y$”

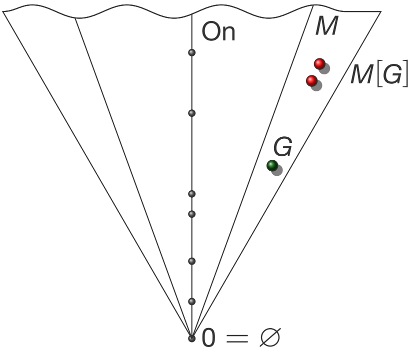
Absoluteness arguments for Turing degrees

(INCOMP):

“$\exists\, x, y$ s.t. $x \not\leqslant_T y \;\&\; x \not\geqslant_T y$”

$2^{\aleph_0} \geqslant \aleph_2 \Rightarrow$ (INCOMP)

Absoluteness arguments for Turing degrees



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$2^{\aleph_0} \geqslant \aleph_2 \Rightarrow$ (INCOMP)

$M[G] \models \mathrm{ZFC} + 2^{\aleph_0} = \aleph_2$

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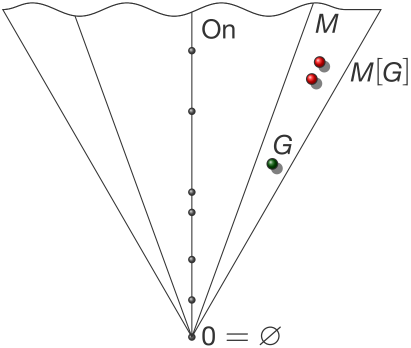

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(INCOMP) is Σ^1_1

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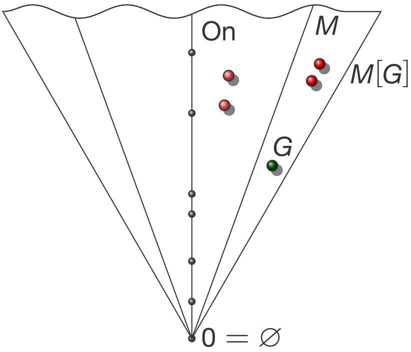

(INCOMP):
"$\exists x, y$ s.t. $x \not\leqslant_T y \;\&\; x \not\geqslant_T y$"

$2^{\aleph_0} \geqslant \aleph_2 \Rightarrow$ (INCOMP)

$M[G] \models \mathrm{ZFC} + 2^{\aleph_0} = \aleph_2$

(INCOMP) is Σ^1_1

(*Levy-Shoenfield*) Σ^1_2 absolute for $\omega_1^M \subseteq N$

Kozen's theory of subrecursive indexings

Axiomatic framework of indexings of closed classes of rec. functions

graph$(U) \equiv^m$ diag, $U \not\leqslant_C$ graph(U)

s-m-n, but no recursion theorem.

Weak recursion theorem:
$\exists f_p \in \Omega$ s.t. $\forall x, y$
$$\varphi_{f_p(x)}(y) = \varphi_x(f_p(x), y)$$

$g \notin C$, g 0-1 valued and comp.,
$\Rightarrow \exists h$ computable s.t. $g = \text{diag}_h$

If $P \neq NP$ is provable, then it is provable by diagonalisation.

closed : π_1, π_2, constant functions, cond, composition, pair.

$f \leqslant^m_C g$ if $\exists h \in C, f = g \circ h$

$f \leqslant_C g$ if $f \in$ smallest class closed $\supseteq \{g\} \cup C$

$\Omega =$ smallest class closed under comp, constant functions, pair

$\leqslant^m = \leqslant^m_\Omega$

$$\text{diag}_h = x \mapsto \begin{cases} 1 & \text{if } \varphi_{h(x)}(x) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Basic ideas behind sub-computabilities

Functions whose graph is enumerated by 1-1 functions from a particular class $\mathbb{C}$.

$\mathbb{C}$ is a class of total recursive functions with an indexing and good closure properties.
(*e.g. primitive recursive or α-recursive functions*)

graph$(\psi) = f(\mathbb{N})$, for $f \in \mathbb{C}$

$\Phi^{\mathbb{C}}_{\cdot}$ enumerates the functions of $\mathbb{C}$.

$\forall n \in \mathbb{N}$, $\mathbb{W}^{\mathbb{C}}_n$ is the induced *r.e.* sets ($\mathbb{C}$-*r.e.* sets).

$\forall n \in \mathbb{N}$, $\varphi^{\mathbb{C}}_n$ is the function of graph $\mathbb{W}^{\mathbb{C}}_n$.

2. sub-computabilities

From total functions to sub-computabilities

A set of total functions with good closure properties.

$\mathbb{c} \subseteq$ total recursive functions
$\mathbb{p} =$ primitive recursive functions
$\mathbb{p} \subseteq \mathbb{c}$

From total functions to sub-computabilities

$\mathbb{C} \subseteq$ total recursive functions
$\mathbb{P} =$ primitive recursive functions
$\mathbb{P} \subseteq \mathbb{C}$

Enumeration $\Phi^{\mathbb{C}}_{\cdot}$.

No *true* $\mathbb{C}$-universal machine.
i.e. $\nexists u_{\mathbb{C}} \in \mathbb{C}, u_{\mathbb{C}}(x, y) = \Phi^{\mathbb{C}}_x(y)$

But a *step-by-step* $\mathbb{C}$-universal machine in $\mathbb{P}$
i.e. $\exists\, \text{sim}_{\mathbb{C}} \in \mathbb{P}$,
$\text{sim}_{\mathbb{C}}(x, y, s) = \Phi^{\mathbb{C}}_x(y)$ for a large enough s

From total functions to sub-computabilities

$\mathbb{C}$-recursively enumerable sets?

W $\mathbb{C}$-r.e if

$\exists f \in \mathbb{C}, W =$ 1-1 prefix of $f(\mathbb{N})$

Natural enumeration:

$\mathbf{W}^{\mathbb{C}}_e =$ 1-1 prefix of $\Phi^{\mathbb{C}}_e(\mathbb{N})$

$\mathbb{C} \subseteq$ total recursive functions

$\mathbb{P} =$ primitive recursive functions

$\mathbb{P} \subseteq \mathbb{C}$

Enumeration $\Phi^{\mathbb{C}}_{.}$ of $\mathbb{C}$

$\mathrm{sim}_{\mathbb{C}}(x, y, s) = \Phi^{\mathbb{C}}_{x,s}(y)$

From total functions to sub-computabilities

Notions of recursive sets?

TFANE:

- W χ-$\mathbb{C}$-rec. if $\chi_W \in \mathbb{C}$;
- W wkly-$\mathbb{C}$-rec. if W and $\overline{W}$ $\mathbb{C}$-*r.e.*;
- W stgly-$\mathbb{C}$-*r.e.* if W is $\mathbb{C}$-*r.e.* by an increasing function.

$\mathbb{C} \subseteq$ total recursive functions
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Enumeration $\Phi^{\mathbb{C}}_{.}$ of $\mathbb{C}$
$\mathrm{sim}_{\mathbb{C}}(x, y, s) = \Phi^{\mathbb{C}}_{x,s}(y)$

W $\mathbb{C}$-r.e if $\exists f \in \mathbb{C}$, 1-1, $f(\mathbb{N}) = W$

From total functions to sub-computabilities

Partial functions?

ψ somewhat-$\mathbb{C}$-rec. if $Gr(\psi)$ $\mathbb{C}$-*r.e.*
$\mathfrak{S}_{\mathbb{C}} = \{\psi : \psi \text{ somewhat-}\mathbb{C}\text{-rec.}\}$

Natural enumeration :
$Gr(\varphi_e^{\mathbb{C}}) = \mathbb{W}_e^{\mathbb{C}}$

$\mathbb{C} \subseteq$ total recursive functions
$\mathbb{P} =$ primitive recursive functions
$\mathbb{P} \subseteq \mathbb{C}$

Enumeration $\Phi_{\cdot}^{\mathbb{C}}$ of $\mathbb{C}$
$\text{sim}_{\mathbb{C}}(x, y, s) = \Phi_{x,s}^{\mathbb{C}}(y)$

W $\mathbb{C}$-r.e if $\exists f \in \mathbb{C}$, 1-1, $f(\mathbb{N}) = W$

W χ-$\mathbb{C}$-rec. if $\chi_W \in \mathbb{C}$
W wkly-$\mathbb{C}$-rec. if W and $\overline{W}$ $\mathbb{C}$-*r.e.*
W stgly-$\mathbb{C}$-*r.e.* if W is $\mathbb{C}$-*r.e.* increasingly

It works almost as the usual computability

Immediate result:

Heredity theorem [Koz'minyh 72, GL]
If E r.e, W $\mathbb{C}$-r.e, $W \subseteq E$
Then E $\mathbb{C}$-*r.e.*

It works almost as the usual computability

Heredity: E r.e, W $\mathbb{C}$-r.e,
$W \subseteq E \Rightarrow E$ $\mathbb{C}$-*r.e.*

Classical properties:

It works almost as the usual computability

Heredity: E r.e, W $\mathbb{C}$-r.e,
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Kleene's recursion theorem
$\forall f \in \mathbb{C}$, A infinite wkly-$\mathbb{C}$-comp,
$\mathrm{dom}(\varphi_a^{\mathbb{C}}) = A$,
$\exists n\ \varphi_n^{\mathbb{C}}|_{\overline{A}} \cong \varphi_{f(n)}^{\mathbb{C}}|_{\overline{A}}$ and $\varphi_n^{\mathbb{C}}|_A \cong \varphi_a^{\mathbb{C}}$

It works almost as the usual computability

Proof.

Consider the $\mathbb{C}$-comp. function:

$$\psi_x(u) : u \mapsto \begin{cases} \varphi_a^{\mathbb{C}}(u) & \text{if } u \in A \\ \varphi^{\mathbb{C}}_{\varphi_x^{\mathbb{C}}(x)}(u) & \text{otherwise.} \end{cases}$$

of $\mathbb{C}$-index $d_a(x)$, $d_a \in \mathbb{C}$.

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Let e_a be a $\mathbb{C}$-index for $f \circ d_a$.

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of $\mathbb{C}$-index $d_a(x)$, $d_a \in \mathbb{C}$.

Let e_a be a $\mathbb{C}$-index for $f \circ d_a$.

Now we have that, for all u:

$\varphi^{\mathbb{C}}_{d_a(e_a)}(u) \cong \varphi^{\mathbb{C}}_a(u)$ if $u \in A$

$\varphi^{\mathbb{C}}_{d_a(e_a)}(u) \cong \varphi^{\mathbb{C}}_{\varphi^{\mathbb{C}}_{e_a}(e_a)}(u) \cong \varphi^{\mathbb{C}}_{f \circ d_a(e_a)}(u)$ otherwise.

Heredity: E r.e, W $\mathbb{C}$-r.e,

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Kleene's recursion thm

$\forall f \in \mathbb{C}$, A infinite wkly-$\mathbb{C}$-comp,

$\mathrm{dom}(\varphi^{\mathbb{C}}_a) = A$,

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Finally, we choose

$n = d_a(e_a)$. □

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$\forall f \in \mathbb{C}$, A infinite wkly-$\mathbb{C}$-comp,
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It works almost as the usual computability

Myhill isomorphism thm (on $\mathbb{C}$)
$A \equiv_1^{\mathbb{C}} B \iff A \equiv^{\mathbb{C}} B$.

Rogers' isomorphism thm
$\psi_\cdot^{\mathbb{C}}$ acceptable iff
$\exists f \in \mathbb{C}\ \forall e\ \psi_e^{\mathbb{C}}(\cdot) \cong \varphi_{f(e)}^{\mathbb{C}}(\cdot)$

Heredity: E r.e, W $\mathbb{C}$-r.e,
$W \subseteq E \Rightarrow E$ $\mathbb{C}$*-r.e.*

Kleene's recursion thm
$\forall f \in \mathbb{C}$, A infinite wkly-$\mathbb{C}$-comp,
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Creativity / Productivity notion

Heredity: E r.e, W $\mathbb{C}$-r.e,
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$\forall f \in \mathbb{C}$, A infinite wkly-$\mathbb{C}$-comp,
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Myhill isomorphism on $\mathbb{C}$:
$A \equiv^{\mathbb{C}}_1 B \iff A \equiv^{\mathbb{C}} B$.

Rogers' isomorphism:
ν acceptable iff
$\exists f \in \mathbb{C}\ \forall e\ \nu(e,\cdot) \cong \mathbf{U}_{\mathbb{C}}(f(e),\cdot)$

Sub-reducibilities

Remember the three recursivity notions

W χ-$\mathbb{C}$-rec. if $\chi_W \in \mathbb{C}$

W wkly-$\mathbb{C}$-rec. if W and $\overline{W}$ $\mathbb{C}$-*r.e.*

W stgly-$\mathbb{C}$-*r.e.* if W is $\mathbb{C}$-*r.e.* by an increasing function

Sub-reducibilities

W χ-$\mathbb{C}$-rec. if $\chi_W \in \mathbb{C}$

$A \leqslant^{\chi}_{\mathbb{C}\text{-T}} B$ if $\chi_A \in \mathbb{C}[\chi_B]$

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W wkly-$\mathbb{C}$-rec. if W and $\overline{W}$ $\mathbb{C}$-*r.e.*

$A \leqslant^{w}_{\mathbb{C}\text{-T}} B$ if $e_A, e_{\overline{A}} \in \mathbb{C}[e_B, e_{\overline{B}}]$

W stgly-$\mathbb{C}$-*r.e.* if W is $\mathbb{C}$-*r.e.* by an increasing function

Sub-reducibilities

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W stgly-$\mathbb{C}$-*r.e.* if W is $\mathbb{C}$-*r.e.* by an increasing function

$A \leqslant^{s}_{\mathbb{C}\text{-T}} B$ if $p_A, p_{\overline{A}} \in \mathbb{C}[p_B, p_{\overline{B}}]$

New versions of usual sets and functions

Diagonal set: $\mathcal{K} = \{e : \varphi_e(e)\downarrow\}$

φ^{C}-Diag. set: $\mathcal{K}_{\mathrm{C}} = \{e : \varphi_e^{\mathrm{C}}(e)\downarrow\}$

Φ^{C}-Diag. set: $\mathcal{K}_{\mathrm{C}}^{\Phi} = \{e : \Phi_e^{\mathrm{C}}(e) > 0\}$

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$A \leqslant^{\chi}_{\mathrm{C}\text{-T}} B$ if $\chi_A \in \mathrm{C}[\chi_B]$

W wkly-C-rec. if W and $\overline{W}$ C-*r.e.*

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Diagonal set: $\mathcal{K} = \{e : \varphi_e(e)\downarrow\}$

$\varphi^{\mathbb{C}}$-Diag. set: $\mathcal{K}_{\mathbb{C}} = \{e : \varphi^{\mathbb{C}}_e(e)\downarrow\}$

$\Phi^{\mathbb{C}}$-Diag. set: $\mathcal{K}^{\Phi}_{\mathbb{C}} = \{e : \Phi^{\mathbb{C}}_e(e) > 0\}$

$\mathcal{K}$ and $\mathcal{K}_{\mathbb{C}}$ are $\mathbb{P}$-*r.e.* and Turing-complete.

$\mathcal{K}^{\Phi}_{\mathbb{C}}$ is wkly-$\mathbb{P}$-computable.

W χ-$\mathbb{C}$-rec. if $\chi_W \in \mathbb{C}$

$A \leqslant^{\chi}_{\mathbb{C}\text{-T}} B$ if $\chi_A \in \mathbb{C}[\chi_B]$

W wkly-$\mathbb{C}$-rec. if W and $\overline{W}$ $\mathbb{C}$-*r.e.*

$A \leqslant^{w}_{\mathbb{C}\text{-T}} B$ if $e_A, e_{\overline{A}} \in \mathbb{C}[e_B, e_{\overline{B}}]$

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Φ^{c}-Diag. set: $\mathcal{K}^{\Phi}_{\mathsf{c}} = \{e : \Phi^{\mathsf{c}}_e(e) > 0\}$

$\mathcal{K}$ and $\mathcal{K}_{\mathsf{c}}$ are p-*r.e.* and Turing-complete.

$\mathcal{K}^{\Phi}_{\mathsf{c}}$ is wkly-p-computable.

$\mathcal{K}^{\Phi}_{\mathsf{c}}$ is recursive.

W χ-c-rec. if $\chi_W \in \mathsf{c}$

$A \leqslant^{\chi}_{\mathsf{c}\text{-T}} B$ if $\chi_A \in \mathsf{c}[\chi_B]$

W wkly-c-rec. if W and $\overline{W}$ c-*r.e.*

$A \leqslant^{w}_{\mathsf{c}\text{-T}} B$ if $e_A, e_{\overline{A}} \in \mathsf{c}[e_B, e_{\overline{B}}]$

New versions of usual sets and functions

Diagonal set: $\mathcal{K} = \{e : \varphi_e(e) \downarrow\}$

φ^{C}-Diag. set: $\mathcal{K}_{\mathrm{C}} = \{e : \varphi_e^{\mathrm{C}}(e) \downarrow\}$

Φ^{C}-Diag. set: $\mathcal{K}_{\mathrm{C}}^{\Phi} = \{e : \Phi_e^{\mathrm{C}}(e) > 0\}$

$\mathcal{K}$ and $\mathcal{K}_{\mathrm{C}}$ are P-*r.e.* and Turing-complete.

$\mathcal{K}_{\mathrm{C}}^{\Phi}$ is wkly-P-computable.

$\mathcal{K}_{\mathrm{C}}^{\Phi}$ is recursive.

$\mathcal{K}_{\mathrm{C}}^{\Phi}$ is χ-C-intermediate.

W χ-C-rec. if $\chi_W \in \mathrm{C}$

$A \leqslant^{\chi}_{\mathrm{C}\text{-T}} B$ if $\chi_A \in \mathrm{C}[\chi_B]$

W wkly-C-rec. if W and $\overline{W}$ C-*r.e.*

$A \leqslant^{w}_{\mathrm{C}\text{-T}} B$ if $e_A, e_{\overline{A}} \in \mathrm{C}[e_B, e_{\overline{B}}]$

A parenthesis on refined degree structure

Each Turing degree is divided in infinitely many $\mathbb{C}$-degrees.

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Each *r.e.* Turing degree contains infinitely many $\mathbb{C}$-*r.e.* degrees.

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$\mathbb{C}$ reducibilities create objects in the recursive world.

Fine structure of degrees.

A parenthesis on refined degree structure

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Fine structure of degrees.

Just as Kristiansen's *honest elementary degrees.*

honest: monotone, dominates 2^n and has elementary graph.

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Each Turing degree is divided in infinitely many $\mathbb{C}$-degrees.

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$\mathbb{C}$ reducibilities create objects in the recursive world.

Fine structure of degrees.

Just as Kristiansen's *honest elementary degrees.*

Honest ϵ_0-elementary degrees have minimal pairs.

honest: monotone, dominates 2^n and has elementary graph.

an honest function g is ϵ_0-elementary in an honest function f iff PA $+ \text{Tot}(f) \vdash \text{Tot}(g)$

Growth speed of functions and χ-jumps

Ack is **not** $\mathfrak{p}$-fundamental.

Ack is χ-$\mathfrak{p}$-complete.

Ack$(\mathbb{N})$ is **not** somewhat $\mathfrak{p}$-comp.

	$\mathfrak{p}$-r.e	w-$\mathfrak{p}$-rec	χ-$\mathfrak{p}$-rec
$\neg\mathfrak{p}$-r.e	-	-	$A_{ck}(\mathbb{N})$
$\neg$w-$\mathfrak{p}$-rec	$\mathrm{K}_{\mathfrak{p}}$	-	$A_{ck}(\mathbb{N})$
$\neg\chi$-$\mathfrak{p}$-rec	$\mathrm{K}_{\mathfrak{p}}^{\Phi}$	$\mathrm{K}_{\mathfrak{p}}^{\Phi}$	-

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A natural extension: $\mathfrak{p}[\text{Ack}]$

	$\mathfrak{p}$-r.e	w-$\mathfrak{p}$-rec	χ-$\mathfrak{p}$-rec
$\neg\mathfrak{p}$-r.e	-	-	$A_{ck}(\mathbb{N})$
$\neg$w-$\mathfrak{p}$-rec	$K_{\mathfrak{p}}$	-	$A_{ck}(\mathbb{N})$
$\neg\chi$-$\mathfrak{p}$-rec	$K_{\mathfrak{p}}^{\Phi}$	$K_{\mathfrak{p}}^{\Phi}$	-

Growth speed of functions and χ-jumps

Ack is **not** $\mathbb{p}$-fundamental.
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Ack$(\mathbb{N})$ is **not** somewhat $\mathbb{p}$-comp.

A natural extension: $\mathbb{p}[\text{Ack}]$

For some $\mathbb{C}$, one can find a g_γ that grows faster than functions in $\mathbb{C}$

	$\mathbb{C}$-r.e	w-$\mathbb{C}$-rec	χ-$\mathbb{C}$-rec
$\neg\mathbb{C}$-r.e	-	-	$g_\gamma(\mathbb{N})$
$\neg$w-$\mathbb{C}$-rec	$\mathcal{K}_{\mathbb{C}}$	-	$g_\gamma(\mathbb{N})$
$\neg\chi$-$\mathbb{C}$-rec	$\mathcal{K}_{\mathbb{C}}^{\Phi}$	$\mathcal{K}_{\mathbb{C}}^{\Phi}$	-

With g_γ a recursive function that grows faster than functions in $\mathbb{C}$

Growth speed of functions and χ-jumps

Busy beaver functions:

$\mathbb{BB}_{\mathbb{C}}(x) = \max\{\varphi_i^{\mathbb{C}}(0) : i \leqslant x\}$

$\mathbb{BB}_{\mathbb{C}}^{\Phi}(x) = \max\{\Phi_i^{\mathbb{C}}(0) : i \leqslant x\}$

$\mathbb{BB}_{\mathbb{C}}^{\Phi}$ is **not** $\mathbb{C}$-fundamental.

$\mathbb{BB}_{\mathbb{C}}^{\Phi}$ is χ-$\mathbb{C}$-complete.

$\mathbb{BB}_{\mathbb{C}}^{\Phi}(\mathbb{N})$ is **not** somewhat $\mathbb{C}$-comp.

We denote by ⓒ the sub-computability of foundation $\mathbb{C}[\mathbb{BB}_{\mathbb{C}}^{\Phi}]$.

	$\mathbb{C}$-r.e	w-$\mathbb{C}$-rec	χ-$\mathbb{C}$-rec
$\neg\mathbb{C}$-r.e	-	-	$\mathbb{BB}_{\mathbb{C}}^{\Phi}(\mathbb{N})$
$\neg$w-$\mathbb{C}$-rec	$\mathcal{K}_{\mathbb{C}}$	-	$\mathbb{BB}_{\mathbb{C}}^{\Phi}(\mathbb{N})$
$\neg\chi$-$\mathbb{C}$-rec	$\mathcal{K}_{\mathbb{C}}^{\Phi}$	$\mathcal{K}_{\mathbb{C}}^{\Phi}$	-

With g_γ a recursive function that grows faster than functions in $\mathbb{C}$

3. beyond ω_1^{ck} !

Higher recursion theory (Kripke,Kreisel,Sacks,Platek)

classical recursion theory lifted from $\mathbb{N}$

admissibles, α-recursion theory

$\alpha \times \cdots \times \alpha \leftrightarrow \alpha$
$\alpha \leftrightarrow L_\alpha$

$\exists$ eff. enum.: $\alpha \longrightarrow \alpha$-finite sets
$\alpha \longrightarrow \alpha$-*r.e.* sets

A is α-*r.e.* iff $A = \mathrm{rg}(f)$, f α-finite

$A \subseteq \mathbb{N}$ *r.e.* iff $\Sigma_1(H_\omega = L_\omega, \in)$

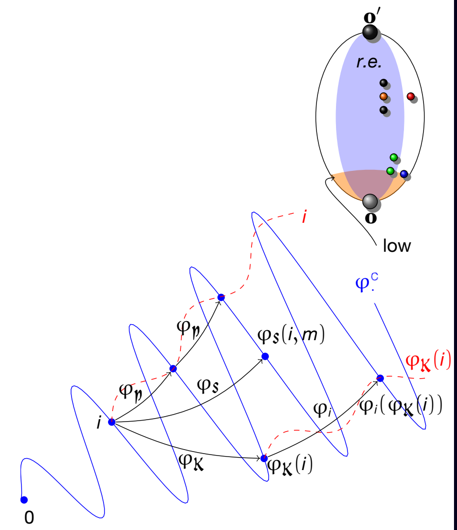
$A \subseteq \alpha$ α-*r.e.* if $\Sigma_1(L_\alpha, \in)$

α admissible *iff* limit and
$\nexists \gamma < \alpha\ \exists \alpha$-rec. f. from γ onto α

$x \subseteq \alpha$ α-finite if $x \in L_\alpha$

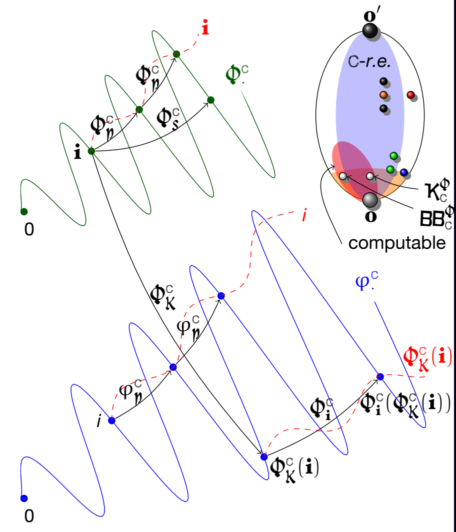
5. summary

Someone uttered the word “*computability*”



$\mathfrak{W}_\cdot\ (\varphi_\cdot)$	
padding $\mathfrak{p}$	$\varphi_i \cong \varphi_{\varphi_{\mathfrak{p}}(i)}$
s-m-n $\mathfrak{s}$	$\varphi_i(\langle m,x\rangle) \cong \varphi_{\varphi_{\mathfrak{s}}(i,m)}(x)$
Kleene $\mathfrak{K}$	$\varphi_{\varphi_{\mathfrak{K}}(i)} \cong \varphi_{\varphi_i(\varphi_{\mathfrak{K}}(i))}$
Rice, Rogers	creativity, $\varphi_\cdot \leftrightarrow \psi_\cdot$
$\mathfrak{K} = \{x : \varphi_x(x)\downarrow\}$	r.e. *m*-complete
$\mathbb{BB}$ (Busy Beaver)	¬ r.e. *m*-complete
Friedberg-Muchnik	$\varnothing \prec_T \mathfrak{W}_i \prec_T \varnothing'$
low r.e.	$X' \preccurlyeq_T \varnothing'$
minimal r.e. pair	$X \preccurlyeq_T \mathfrak{W}_i \wedge X \preccurlyeq_T \mathfrak{W}_j \Rightarrow X \preccurlyeq_T \varnothing$

Someone mentioned “*sub-computabilities*”



$\Phi^{c}_{\cdot}$: total rec. functions

$\Phi^{c}_{\cdot}, \mathbb{W}^{c}_{\cdot}$ — c-r.e.: 1-1 enum. by a Φ^{c}_{i}

$\varphi^{c}_{\cdot}$ — c-r.e. graph

Kleene $\mathbb{K}$ — $\Phi^{©}_{\Phi^{c}_{\mathbb{K}}(\mathbf{i})} \simeq \Phi^{c}_{\Phi^{c}_{\mathbf{i}}(\Phi^{c}_{\mathbb{K}}(\mathbf{i}))}$

$\varphi^{c}_{\Phi^{c}_{\mathbb{K}}(\mathbf{i})}|_{D_j} \simeq \varphi^{c}_{\Phi^{c}_{\mathbf{i}}(\Phi^{c}_{\mathbb{K}}(\mathbf{i}))}|_{D_j}$

(co-inf., r.e. $\overline{\mathrm{dom}(\varphi^{c}_{j})} = D_j$) $\varphi^{c}_{\Phi^{c}_{\mathbb{K}}(\mathbf{i})} \simeq \varphi^{c}_{j}$

Rice, Rogers — c-creativity, $\varphi^{c}_{\cdot} \leftrightarrow \psi^{c}_{\cdot}$

$\mathbb{K}_{c} = \{x : \varphi^{c}_{x}(x)\downarrow\}$ — c-r.e. *m*-complete

$\mathbb{K}^{\Phi}_{c} = \{x : \Phi^{c}_{x}(x) > 0\}$ — χ-c-low c-r.e.

$BB_{c}(x) = \max\{\varphi^{c}_{i}(0) : i \leqslant x\}$ — ¬ c-r.e. *m*-complete

$BB^{\Phi}_{c}(x) = \max\{\Phi^{c}_{i}(0) : i \leqslant x\}$ — χ-c-low ¬ c-r.e.

thank you for your attention, ↓