Higher sub-computabilities

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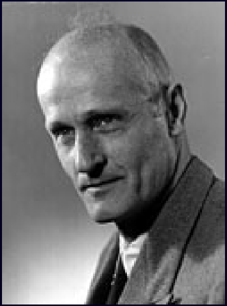
February 22nd 2012

1. computabilities and recursion

below Church-Kleene's ω_1^{CK}



Building recursive functions the ol' way.

Basic functions

- $\mathbf{o}$: constant null function
- $\mathbf{s}$: successor function
- $\langle \cdot, \cdot \rangle$: pairing function
- π_1, π_2: projection functions

Operations on functions

- $\circ$: composition
- μ: recursion operator

Nice algebraic properties are nice.

Indexing

- $\varphi_\cdot$: enumeration of partial functions
- $\varphi_u(\langle e, n\rangle) = \varphi_e(n)$: universal function

Classical results over the indexing

- s-m-n $\exists s, \forall x, y, \varphi_e(y, x) = \varphi_{s(e,y)}(x)$
- Kleene's recursion theorem $\forall f, \exists n, \varphi_{f(n)} \equiv \varphi_n$
- Rogers isomorphism

Classical problems over the indexing

- Halting problem is undecidable
- recursive function "=" Σ_1 formula

Please, draw me a provably total recursive function.

Recursion operators for total functions

- primitive recursive functions $4 \longrightarrow 3 \longrightarrow 2 \longrightarrow 1 \longrightarrow 0$
- α-recursive operator (go down along α) $\omega^2 \longrightarrow \omega + \omega \longrightarrow \cdots \longrightarrow 0$

Fast growing functions

- Ackermann function
- Goodstein sequences, etc. $5 = 2^2 + 1 \longrightarrow 3^3 + 1 - 1 \longrightarrow \cdots \longrightarrow 0$

THM **Rathjen** Provable total functions of $\mathsf{PA} + \cup_{\alpha \in A}(\mathsf{TI}(\alpha))$ are exactly the α-recursive functions, for $\alpha \in A$

What if our base functions are such a set of recursive functions?

2. sub-computabilities

slices of ω_1^{CK}

From indexings of total recursive functions to…what?

$\phi^{\mathbb{C}}_{\cdot}$: enumeration of total functions

- well closed class (composition, basic functions, primitive recursion)
- s-m-n, no Kleene's recursion theorem
- no universal function **in the class**

$W^{\mathbb{C}}_{\cdot}$: enumeration of one-one recursively enumerated sets

- all $\mathbb{C}$-r.e. sets are r.e.
- finite sets are $\mathbb{C}$-r.e.

$\varphi^{\mathbb{C}}_{\cdot}$: enumeration of partial functions of graph $W^{\mathbb{C}}_{\cdot}$

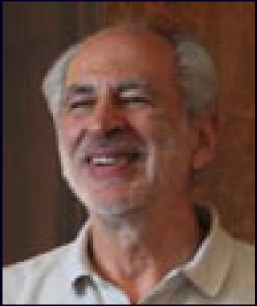
- "Half-Kleene's recursion theorem"
- $\mathbb{C}$-computable functions are computable

Refinements of Turing reduction between sets and degrees

- $\mathbb{C}$-intermediate sets, $\mathbb{C}$-complete sets, $\mathbb{C}$-low sets, etc.

3. higher sub-computabilities

beyond ω_1^{CK}



Higher Recursion Theory in a nutshell

Informally

- ω_1^{CK} is now "recursive" from our point of view

Less informally

- α: admissible ordinal *(stable by all functions definable inside it)*
- L_α: α^{th} level of Godel's constructible hierarchy
- $\Sigma_1(L_\alpha)$: objects of the higher computability

Basic results

- s-m-n
- recursion theorem *à la* Kleene
- behave as a computability

Higher Sub-Computabilities

Higher computability fits in our framework

- Enumeration of basic objects ($\alpha \longrightarrow L_\alpha$)
- Enumeration of partial functions *computable* from these objects

Generalization to non-admissible ordinals

Sub-computabilities do not require the ordinal to be admissible.

Work in progress

- identify properties of higher sub-computabilities associated to particular admissible ordinals
- identify links beteen higher sub-computabilities and sub-computabilities

Thank you for your attention!