Towards a fine structure of computabilities

Vers une structure fine des calculabilités

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Motivations

Computational models for fragments of computability

A computability framework for subrecursion

A hierarchy of computabilities, from primitive recursion to admissible recursion, and above

1. classical computability

Church, Kleene, Rosser, Turing

Computable functions

Basic operations

$\mathbf{o} : x \mapsto 0$

$\mathbf{s} : x \mapsto x + 1$

$\mathbf{cond} : x, y, a, b \mapsto \begin{cases} a & \text{if } x = y \\ b & \text{otherwise} \end{cases}$

Operations can be composed

$\mathbf{s} \circ \mathbf{o} = x \mapsto 1$

Complex operations

μ: minimisation

$\mathbf{rec}_{\mathrm{p}}$: primitive recursion

Computation tree

Not necessarily finite

Computable functions

Basic operations

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μ: minimisation

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Computation tree

Not necessarily finite

Computation Tree / Derivation Tree

$(x \mapsto \mathbf{cond}(\mathbf{s}(x), 3, 9, 7))\ (3)$

$\mathbf{cond}(\mathbf{s}(3), 3, 9, 7)$

$\mathbf{cond}(4, 3, 9, 7)$

7

The more complex the machine, the more complex the tree.

Notations

- Enumeration of all the recursive functions, indexed by the natural integers:

$$(\varphi_e)_{e \in \omega}$$

- Convergence of a function:

$$\varphi_{26}(13) \downarrow = 7$$

- Divergence of a function:

$$\varphi_{34}(4) \uparrow$$

- Equivalence of functions:

$$\varphi_{e_1}(x_1) \simeq \varphi_{e_2}(x_2)$$

if both computations diverge or converge to the same value.

Universality and total functions

- Recursive universal function

$$\exists u, \varphi_u : (\langle e, x \rangle) \mapsto \varphi_e(x)$$

- Functions cannot all be total

$$f : x \mapsto \mathbf{s} \circ \varphi_u \left(\langle x, x \rangle \right)$$

$$\exists e, f = \varphi_e$$

$$\begin{aligned} f(e) &\cong \mathbf{s}(\varphi_u(\langle e, e \rangle)) \\ &\cong \varphi_e(e) + 1 \\ &\cong f(e) + 1 \end{aligned}$$

Canonical form and interpretation

Kleene's Normal Form

There exist an elementary function F and a recursive primitive predicate T such that:

$$\forall e, x,\ \varphi_u(\langle e, x\rangle) \cong \varphi_e(x) \cong F(\mu y.T(e, x, y))$$

- Only μ operator
- Function index
- Computation tree
- Input
- Checker

Halt

Computation tree

Finite?

Bounded?

s_n^m and Fixed Point

- s_n^m:
  There exists a recursive function s_n^m such that $\forall m, n, e$,

$$\varphi_e(\langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle) \cong \varphi_{s_n^m(e,x_1,\ldots,x_n)}(\langle y_1, \ldots, y_m \rangle)$$

- Fixed point:
  For each total recursive function f we can recursively compute an n such that:

$$\forall x, \varphi_n(x) \cong \varphi_{f(n)}$$

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Example

Let f be a function such that $\varphi_{f(n)} : x \mapsto \varphi_n(x) + 1$. Then there is an n verifying:

$$\forall x, \varphi_n(x) \cong \varphi_{f(n)}(x) \cong \varphi_n(x) + 1$$

which is an index for the nowhere defined function.

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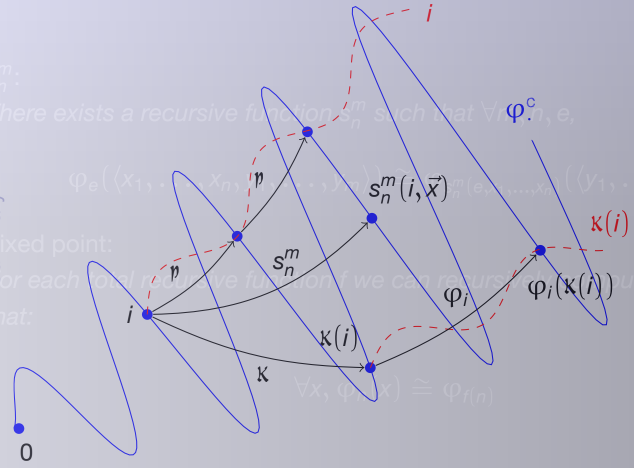

2. from total functions to partial computabilities

An interesting trade-off

Classes of total functions

For a class $\mathcal{C}$ of total recursive functions, closed under composition:

- Universal function not in $\mathcal{C}$
- No unbounded search
- Limited function growth (e.g. Ackermann function not in $\mathcal{P}$)
- Implies limited power

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How to deal with these limitations?

Primitive recursive coding schema

For a class C *with an enumeration* $\varphi^{\mathrm{C}}_{\cdot}$*:*

Indices

Toolbox for indices:

Compute new indices for a function

Compute index of composition

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Requirements:

padding function: $\mathfrak{p}$

$\forall e, (\mathfrak{p}(e) > e) \wedge (\phi^{\mathcal{C}}_{\mathfrak{p}(e)} = \phi^{\mathcal{C}}_{e})$

composition function: $\mathfrak{c}$

$\forall e_1, e_2, \phi^{\mathcal{C}}_{\mathfrak{c}(e_1,e_2)} = \phi^{\mathcal{C}}_{e_1} \circ \phi^{\mathcal{C}}_{e_2}$

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Check the validity of a tree

Bound the tree of a given function

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step-by-step simulation: $\mathrm{sim}_{\mathsf{c}}$

$\forall x, e, \exists n, \phi^{\mathsf{c}}_e(x) = \mathrm{sim}_{\mathsf{c}}(e, x, n)$

cost function: $\mathrm{use}_{\mathsf{c}}$

$\forall x, e, \phi^{\mathsf{c}}_e(x) = \mathrm{sim}_{\mathsf{c}}(e, x, \phi^{\mathsf{c}}_{\mathrm{use}_{\mathsf{c}}(e)}(x))$

Fundamental classes

Def: Fundamental class

Basis functions

Contains all the primitive recursive functions

Coding functions

Primitive recursive coding schema: $\mathbf{rec}_{\mathrm{p}}$

Pairing functions: $\langle\cdot,\cdot\rangle, \pi^2_1, \pi^2_2 \in \mathrm{c}$,

Projection functions: $\forall n_1, n_2, \pi^2_i(\langle n_1, n_2\rangle) = n_i$

Closure

Stable by composition: $\forall f, g \in \mathrm{c}, f \circ g \in \mathrm{c}$.

Stable by primitive recursion: $\forall g, h \in \mathrm{c}, \mathbf{rec}_{\mathrm{p}}(g, h) \in \mathrm{c}$

Enumeration

Tied to an enumeration $\phi^{\mathrm{c}}_{\cdot}$ (recursive) which is not in the class

Classes defined using recursion schemata

Definition

Primitive recursive class: p

Smallest fundamental class (stable by primitive recursion: $\mathbf{rec}_{\mathrm{p}}$)

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Primitive recursion

For $g, h \in \mathrm{p}$,

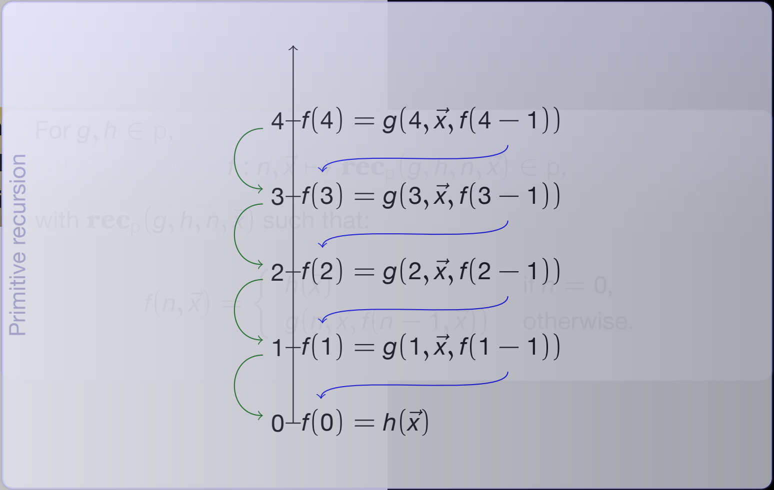
$$f : n, \vec{x} \mapsto \mathbf{rec}_{\mathrm{p}}(g, h, n, \vec{x}) \in \mathrm{p},$$

with $\mathbf{rec}_{\mathrm{p}}(g, h, n, \vec{x})$ such that:

$$f(n, \vec{x}) = \begin{cases} h(\vec{x}) & \text{if } n = 0, \\ g(n, \vec{x}, f(n-1, \vec{x})) & \text{otherwise.} \end{cases}$$

Classes defined using recursion schemata

Definition **Prim**…
Smal…
primi…



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Definition

α-recursive classes: c_{α}
Smallest fundamental class stable by α recursion: $\mathbf{rec}_{\alpha}$

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Definition

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Smallest fundamental class stable by α recursion: $\mathbf{rec}_\alpha$

α-recursion

For $g, h \in c_\alpha$,

$$f : n, \vec{x} \mapsto \mathbf{rec}_{\alpha,\lhd}(g, h, n, \vec{x}) \in c_\alpha$$

with $\mathbf{rec}_{\alpha,\lhd}(g, h, n, \vec{x})$ such that:

$$f(n, \vec{x}) = \begin{cases} g(n, \vec{x}, f(\theta(n, \vec{x}), \vec{x})) & \text{if } \overline{0} \lhd n \text{ and } \theta(n, \vec{x}) \lhd n, \\ h(n, \vec{x}) & \text{otherwise,} \end{cases}$$

where $\overline{\beta}$ stands for the ordinal notation (in $\lhd$) for β.

Classes defined using recursion schemata

Definition

Prim

Smal

primi

Definition

α-re

Smal

α rec

α-recursion

For $g, h \in c_\alpha$,

$$f : n, \vec{x} \mapsto \mathbf{rec}_\alpha(g, h, n, \vec{x}) \in c_\alpha$$

with $\mathbf{rec}_{\alpha,\lhd}(g, h, n, \vec{x})$ such that

$$f(n, \vec{x}) = \begin{cases} g(n, \vec{x}, f(\ldots)) & \text{if } 0 \lhd n \text{ and } \theta(n, \vec{x}) \lhd n, \\ h(n, \vec{x}) & \text{otherwise,} \end{cases}$$

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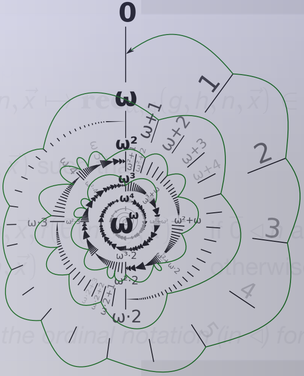

Classes defined using recursion schemata

Definition

Primitive recursive class: p

Smallest fundamental class (stable by primitive recursion: $\mathbf{rec}_p$)

Rathjen: Functions provably total by a theory of ordinal analysis α are exactly α-recursive functions.

Definition

α-recursive classes: c_α

Smallest fundamental class stable by α recursion: $\mathbf{rec}_\alpha$

How far did we get?

Thm s_n^m **for fundamental classes**

Note: Primitive recursion is needed in order to obtain an homogeneous s.

How far did we get?

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s_n^m for fundamental classes

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Thm

Still no recursion theorem

It is actually impossible.

How far did we get?

Thm s_n^m ... fundamental classes

Note: Primitive recursion is needed in order to obtain an homogeneous s.

Thm **Still no recursion theorem**

It is actually impossible.

Kleene and total functions

Recall our previous example:

Let f be a function such that $\varphi_{f(n)} : x \mapsto \varphi_n(x) + 1$. Then there is an n verifying:

$$\forall x, \varphi_n(x) \simeq \varphi_{f(n)}(x) \simeq \varphi_n(x) + 1$$

which is an index of the nowhere defined function.

Such a function cannot be total.

How far did we get?

Thm **s^m_n for fundamental classes**

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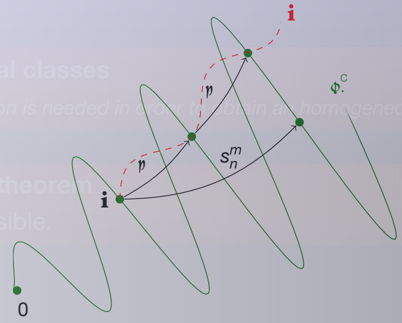
It is actually impossible.

Summary



Rogers' Isomorphism Theorem

Ensures that our results do not depend on our choice of an enumeration

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Go from an acceptable enumeration to another

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Thm

Myhill's isomorphism for fundamental classes

For A and B two sets of integers, f 1-1 from A to B and g 1-1 from B to A, we can build h an isomorphism between A and B.

Rogers' Isomorphism Theorem

Ensures that our results do not depend on our choice of an enumeration

Go from an acceptable enumeration to another

Thm **Myhill's isomorphism for fundamental classes**
For A and B two sets of integers, f 1-1 from A to B and g 1-1 from B to A, we can build h an isomorphism between A and B.

Thm **Rogers' isomorphism for fundamental classes**
Any acceptable enumeration is isomorphic to the canonical one.

Simulation, halt and domination

Primitive recursive case

Simulation, halt and domination

Primitive recursive case

Ackermann function

Definition:

$$A : m, n \mapsto \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{otherwise} \end{cases}$$

Unary version:

$$\text{Ack} : n \mapsto A(n, n)$$

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Ackermann function properties

Grows faster than any primitive recursive function

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Enables us to bound the size of their tree

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Universal simulation with $\mathrm{sim_p}$ and $\mathrm{use_p}$:

$$x \mapsto \mathrm{sim_p}(e, x, \mathrm{Ack}(f(\mathrm{use_p}(e), x)))$$

for some primitive recursive f

Simulation, halt and domination

General case for fundamental classes

Primitive recursive case

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Simulation, halt and domination

General case for fundamental classes

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Busy Beaver function

Definition:

For an enumerable class $\mathcal{C}$ with $\text{sim}_{\mathcal{C}}$ and $\text{use}_{\mathcal{C}}$ functions:

$$\mathbb{BB}^{\Phi}_{\mathcal{C}} = x \mapsto max \left\{ \phi^{\mathcal{C}}_{\text{use}_{\mathcal{C}}(i)}(0) : i \leqslant x \right\}$$

Ackermann function properties

Grows faster than any primitive recursive

Enables us to bound the size of their tree

Universal simulation with sim_p and use_p:

$$x \mapsto \text{sim}_p(e, x, \text{Ack}(f(\text{use}_p(e), x)))$$

for some primitive recursive f

Simulation, halt and domination

General case for fundamental classes

Primitive recursive case

Thm

Busy Beaver properties

Grows faster than any c-fundamental function

Enables us to bound the size of their tree

Universal simulation with sim_c and use_c:

$$x \longmapsto \mathrm{sim}_c(e, x, \mathbb{BB}_c^{\phi}(s_1^1(e, x)))$$

Ackermann function properties

Grows faster than any primitive recursive function

Enables us to bound the size of their tree

Universal simulation with sim_p and use_p:

$$x \longmapsto \mathrm{sim}_p(e, x, \mathrm{Ack}(f(\mathrm{use}_p(e), x)))$$

for some primitive recursive f

A recursive jump for fundamental classes

$\mathbb{BB}_{\mathbb{C}}^{\Phi}$ allows us to totally compute any function in $\mathbb{C}$.

Similar to the classical halting problem.

A recursive jump for fundamental classes

$\mathbb{BB}^{\phi}_{\mathrm{C}}$ allows us to totally compute any function in C.

Similar to the classical halting problem.

Def: Jump

Jump of C: $© = \mathrm{C}[\mathbb{BB}^{\phi}_{\mathrm{C}}]$

Still a fundamental class

$\phi^{©}_{.}$ has a universal function for C.

A recursive jump for fundamental classes

$\mathbb{BB}_{\mathbb{C}}^{\phi}$ allows us to totally compute any function in $\mathbb{C}$.

Similar to the classical halting problem.

Def: Jump

Jump of $\mathbb{C}$: $© = \mathbb{C}[\mathbb{BB}_{\mathbb{C}}^{\phi}]$

Still a fundamental class

$\phi_{\cdot}^{©}$ has a universal function for $\mathbb{C}$.

Relativisations of Kleene's $\mathcal{O}$ and Hyperarithmetic sets

A notion of $\mathbb{C}$-recursive orders (ordinals)

A fine hierarchy of $\mathbb{C}$-degrees

3. c-enumerability and c-recursivities

Complexity of sets

Enumerability and repetitions

Aim: Capture a class complexity through its enumerable sets

Enumerability and repetitions

Aim

Capture a class complexity through its enumerable sets

Enumerable sets and p

Every enumerable set is enumerable by a primitive recursive function.

Enumerate $\mathbb{W}_e$

Simulate φ_e using bounded μ in Kleene's Normal Form:

$$\varphi_{e,s}(x) \simeq F(\mu y \leqslant s . T(e,x,y)) \simeq \mathrm{sim}_T(e,x,s)$$

Enumeration with repetitions:

$$\varphi_e(0), \varphi_e(0), \ldots, \varphi_e(0), \varphi_e(1), \varphi_e(2), \varphi_e(2), \ldots$$

Enumerability and repetitions

Aim

Capture a class complexity through its enumerable sets

Anoyance

Classical definition not interesting

Enumerable sets are c-enumerable.

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Classical characterisations

Domain of a partial function

Range of a partial function

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Enumerable sets are c-enumerable.

Solution: A set is c-enumerable if:

it is finite

or it is the range of a 1-1 $f \in \mathrm{c}$.

Enumeration: $\left(\mathrm{w}_e^{\mathrm{c}}\right)_{e\in\omega}$

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Tips & Tricks

How do we know if a function is 1-1?

We do not.

Check 1-1-ness for each new value

If not, the 1-1 prefix defines a finite set

c-recursivities

Extend the notion of c-enumerability to a notion of c-recursivity.

$\mathbb{C}$-recursivities

Extend the notion of $\mathbb{C}$-enumerability to a notion of $\mathbb{C}$-recursivity.

Classical characterisations

A set E is recursive if:

Its characteristic function χ_E is recursive

E and $\overline{E}$ are enumerable

E can be enumerated increasingly

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χ-c-recursivity Def

c-fundamental characteristic function

weak-c-recursivity Def

c-enumerable and co-c-enumerable

strong-c-enumerability Def

c-enumerable increasingly

strong-c-recursivity Def

Strongly c-enumerable and co-strongly c-enumerable

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χ-c-recursivity (Def)

c-fundamental characteristic function

weak-c-recursivity (Def)

c-enumerable and co-c-enumerable

strong-c-enumerability (Def)

c-enumerable increasingly

strong-c-recursivity (Def)

Strongly c-enumerable and co-strongly c-enumerable

These notions are all different, and all compatible with the classical one (Thm)

Noticeable sets and recursive properties

Def

Diagonal set

$\kappa_{c}^{\phi} = \{e : \phi_{e}^{c}(e) > 0\}$

Noticeable sets and recursive properties

Def — Diagonal set

$\kappa_{\mathrm{c}}^{\phi} = \{e : \phi_e^{\mathrm{c}}(e) > 0\}$

Def — Unary Ackermann range

$A = \mathrm{range}(\mathrm{Ack})$

Def — Busy Beaver range

$B_{\mathrm{p}} = \mathrm{range}(\mathrm{BB}_{\mathrm{p}}^{\phi})$

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Noticeable sets and recursive properties

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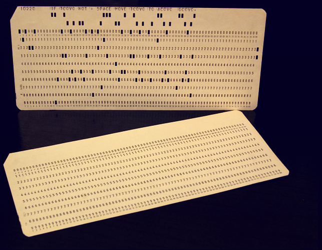
Thm

	r.e.	co-*r.e.*	p-e	s-p-e	co-p-e	co-s-p-e	w-p-rec	s-p-rec	χ-p-rec
κ	✓	✗	✓	✗	✗	✗	✗	✗	✗
$\kappa_{\mathrm{p}}^{\phi}$	✓	✓	✓	✗	✓	✗	✓	✗	✗
A	✓	✓	✗	✗	✓	✓	✗	✗	✓
B_{p}	✓	✓	✗	✗	✓	✓	✗	✗	✓

	r.e.	co-*r.e.*	c-e	s-c-e	co-c-e	co-s-c-e	w-c-rec	s-c-rec	χ-c-rec
κ	✓	✗	✓	✗	✗	✗	✗	✗	✗
$\kappa_{\mathrm{c}}^{\phi}$	✓	✓	✓	✗	✓	✗	✓	✗	✗
B_{c}	✓	✓	✗	✗	✓	✓	✗	✗	✓

4. subcomputabilities

Computability, with holes



c-partial functions

A partial function is a function whose graph is enumerable.

A graph of a function is a well-formed set of integers.

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A graph of a function is a well-formed set of integers.

Def

$\forall e, \mathcal{G}_e$ is the graph induced by $\mathtt{w}_e^{\mathrm{c}}$

Enumeration: $(\varphi_e^{\mathrm{c}})_{e \in \omega}$

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A partial function is a function whose graph is enumerable.

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Def

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Enumeration: $(\varphi_e^{\mathrm{c}})_{e\in\omega}$

Thm

Fundamental functions are c-partials

c-partial functions

A partial function is a function whose graph is enumerable.

A graph of a function is a well-formed set of integers.

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Enumeration: $(\varphi_e^{\mathrm{c}})_{e \in \omega}$

Thm

Fundamental functions are c-partials

Thm

Growth speed is dominated by fundamental functions

c-partial functions

Unusual closure

Thm

Not stable by composition

No s_n^m theorem

A partial function is a function whose graph is enumerable.

A graph of a function is a well-formed set of integers.

$\forall e, \mathcal{G}_e$ is the graph induced by $\mathtt{w}_e^{\mathrm{c}}$

Enumeration: $(\varphi_e^{\mathrm{c}})_{e\in\omega}$

Def

Fundamental functions are c-partials

Thm

Growth speed is dominated by fundamental functions

Thm

C-partial functions

Thm — Unusual closure

Not stable by composition

No s_n^m theorem

Counter examples — No composition

C contains

$$f : n \mapsto \begin{cases} BB_C^{\Phi}(p) & \text{if } n = 2p \\ p & \text{if } n = 2p+1 \end{cases}$$

and $g : n \mapsto 2n$, but not $f \circ g = BB_C^{\Phi}$.

No s_n^m theorem

C contains $f : \langle e, x \rangle \mapsto \varphi_e(x)$ but not all the recursive functions.

A partial function is a function whose graph is enumerable.

A graph of a function is a well-formed set of integers.

Def — $\forall e, \mathcal{G}_e$ is the graph induced by φ_e^C

Enumeration: $(\varphi_e^C)_{e \in \omega}$

Thm — ... partials

Thm — Growth speed is dominated by fundamental functions

c-partial functions

Thm
Unusual closure
Not stable by composition
No s_n^m theorem

Thm
Non-trivial c-creativity/productivity notion

A partial function is a function whose graph is enumerable.

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Def
$\forall e, \mathcal{G}_e$ is the graph induced by $\mathbf{w}_e^{\mathrm{c}}$

Enumeration: $(\varphi_e^{\mathrm{c}})_{e\in\omega}$

Thm
Fundamental functions are c-partials

Thm
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c-partial functions

Thm

Unusual closure

Not stable by composition

No s_n^m theorem

Thm

Non-trivial c-creativity/productivity notion

Thm

Partial Kleene's second recursion theorem for a fundamental class c

For $f \in \mathsf{c}$ and h c-partial of domain A co-enumerable,

$\exists n$ s.t. $\left(\varphi_n^{\mathsf{c}}\right)\restriction_{\overline{A}} \cong \left(\varphi_{f(n)}^{\mathsf{c}}\right)\restriction_{\overline{A}}$ and $\left(\varphi_n^{\mathsf{c}}\right)\restriction_A \cong h$

A partial function is a function whose graph is enumerable.

A graph of a function is a well-formed set of integers.

Def

$\forall e$, $\mathcal{G}_e$ is the graph induced by $\mathbf{w}_e^{\mathsf{c}}$

Enumeration: $\left(\varphi_e^{\mathsf{c}}\right)_{e \in \omega}$

Thm

Fundamental functions are c-partials

Thm

Growth speed is dominated by fundamental functions

c-partial functions

Thm **Unusual closure**
Not stable by composition
No s^m_n theorem

Thm **Non-trivial c-creativity/productivity notion**

Thm **Partial Kleene's second recursion theorem for a fundamental class c**

For $f \in$ c and h c-partial of domain A co-enumerable,

$\exists n$ s.t. $(\varphi^c_n)_{|\overline{A}} \simeq (\varphi^c_{f(n)})_{|\overline{A}}$ and $(\varphi^c_n)_{|A} \simeq h$

A partial function is a function whose graph is enumerable.

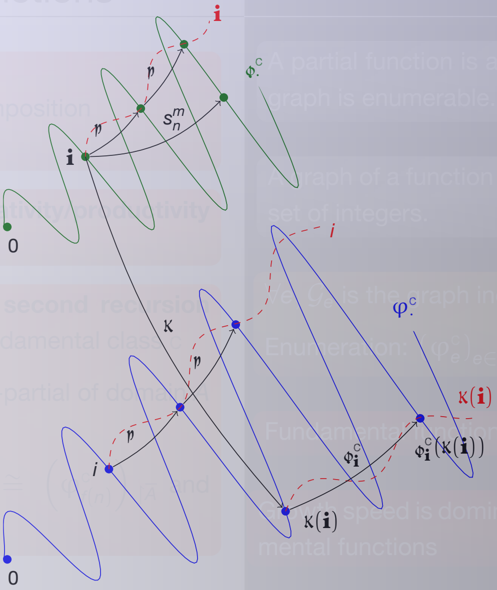
A graph of a function is a well-formed set of integers.

Def $\mathcal{G}_e$ is the graph induced by w^c_e
Enumeration: $(\varphi^c_e)_{e\in\omega}$

Thm Fundamental functions are c-partials

Thm Growth speed is dominated by fundamental functions

Summary



5. fragments of admissible recursion

Rising above

Fragments above computability

The case of Σ-recursion
Functions over sets in admissible levels of Gödel's L hierarchy

An enumeration $\left(\phi_e^{\mathbb{A}}\right)_{e\in\alpha}$ of Δ_0 (α-finite) sets
Plays the role of fundamental functions

An enumeration $\left(\varphi_e^{\mathbb{A}}\right)_{e\in\alpha}$ of Σ_1 (α-enumerable) sets
Plays the role of partial functions

Preliminary results
s_n^m-like theorem
Fixed-point theorem

Perspectives and conclusion

A general computability framework
for studying subrecursion and beyond

Perspectives and conclusion

A general computability framework

for studying subrecursion and beyond

Applications

Relativised notion of Kolmogorov complexity

General fine structure

Study of the $\mathbb{c}$-recursive ordinals

Ordinal iterations of the jump

Proof theory

Links between $\mathbb{c}$-degrees and honest degrees

Yielding results about minimal independent statements using $\mathbb{c}$ classes?

↓

Thank you for your attention.